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En-Route Optimal Flight Planning Constrained to Pass Through Waypoints using MINLP

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***Abstract*—In this paper we study the en-route strategic flight planning of a commercial aircraft constrained to pass through a set of waypoints whose sequence is not predefined. This problem has been solved as an hybrid optimal control problem in which, given the dynamic model of the aircraft, the initial and final states, the path constraints constituting the envelope of flight, and a set of waypoints in the European air space, one has to find the control inputs, the switching times, the optimal sequence of waypoints and the corresponding trajectory of the aircraft that minimize the direct operating cost during the flight. The complete layout of waypoints in the European airspace is reduced and waypoints are gathered into a small number of clusters. The aircraft is constrained to pass through one waypoint inside every cluster of waypoints. The presence of multi point constraints makes the optimal control problem particularly difficult to solve. The hybrid optimal control problem is converted into a mixed integer non linear programming problem first making the unknown switching times part of the state, then introducing binary variable to enforce the constraint of passing through one waypoint inside every cluster, and finally applying a direct collocation method. The resulting mixed integer non linear programming problem has been solved using a branch and bound algorithm. The cases studied and the numerical results show the effectiveness, efficiency and applicability of this method for en-route strategic flight plans definition.**

***Index Terms*—Air Traffic Management, 4D Trajectory Planning, Hybrid Optimal Control, MINLP.**

I. Introduction

The SESAR concept of operations requires a paradigm shift [1], [2] from a highly structured and fragmented system, heavily reliant on tactical decision and with few strategic planning functions, to an integrated one based on collaborative strategic management of trajectories. In the future European ATM system to be built under SESAR, the trajectory becomes the masterpiece of a new set of operating procedures referred to as Trajectory-Based Operations (TBO) [3]. Therefore, the strategic-level implementation of optimal 4D trajectories must be done within a framework of increasing complexity [4]. Thus, 4D trajectory planning and optimization plays a crucial role in the new ATM concept.

Current flight plans are defined according to rigid constraints; in particular, the en-route flight plan portion must specify a certain number of waypoints through which the aircraft must fly, giving expected overfly times and, generally, performing a steady flight, i.e., at constant altitude and velocity, in some defined airways connecting waypoints. Defining, thus, the most efficient flight plan is not easy since there are thousands of waypoints, flight regions with different associated overfly costs, and wind influence to be considered. Moreover it has also been shown that cruising in steady flight is far from an efficient performance [5].

Consequently, with the aim of defining more efficient (en-route) flight plans, this paper presents a MINLP approach for commercial aircrafts strategic horizontal trajectory planning towards TBO.

Some prior research works on aircraft trajectory optimization using optimal control have been presented in [6], [5], and [7]. Other works on trajectory optimization were based on hybrid optimal control in which the sequence of phases was predefined. Problems with known phase sequence have been frequently solved in aerospace engineering as multi-phase problems [8], [9], [10] [11] [12], the last two considering a fixed sequence of waypoints. Another approach to solve multi-phase problems applied to aircraft trajectory optimization was presented in [13], [14], [15], in which a method to generate optimal vertical and 3D flights plans have been presented for commercial aircrafts with a fixed sequence of phases consisting in different operational procedures and aerodynamic configurations.

However, to the best knowledge of the author, the problem of considering multiple waypoints without specifying the sequence has not been studied yet. Thus, the main contribution of this paper is to present an approach based on MINLP to solve a hybrid optimal control problem with non-defined sequence of phases and to apply it to the problem of strategic flight plan definition.

MINLP is the mathematical problem of minimizing a function in a feasible region described as the intersection of a non-linear set and integrity requirements. In the aircraft motion problem herein presented, the MINLP arises from minimizing the objective function of the system subject to the physical constraints. These physical constraints combines non-linear constraints expressing the dynamics of the trajectory of the aircraft and integer requirements to model the choice of the optimal sequence of points. In all generality MINLP is an undecidable problem [16] but if one assumes that the feasible region is bounded, which is the case here, it is NP-Hard. Although bounded MINLPs can be solved in theory, they remain one of the most challenging problems in computational optimization. This is true in particular, when the non-linear set is not convex as it is the case here. Several computer programs have been developed to solve such MINLPs. Among the most efficient codes are the commercial code Baron [17] and the open-source solver Couenne [18]. Unfortunately, as problems size grows above a few hundred variables, these solvers rapidly become impractical. Therefore, we revert to a heuristic approach (one that does not guarantee that the solution found is the best possible). Our approach is based on a nonlinear solver which is able to find locally optimal solution to the problem where the integrity requirements on the variables is removed and a branch-and-bound scheme aimed at finding feasible value for the integer variables. This approach is more commonly followed for problems where the nonlinear region is convex (in such cases it is an exact algorithm) but it has also been found useful for finding good solutions to problems with non-convex nonlinear regions [19].

In the problem formulation, the coupling of the discrete waypoints with the continuous aircraft dynamics results in a hybrid system, i.e., systems that combine a discrete and a continuous dynamics. Some works that model aircraft flights as hybrid systems are [20], [21] and [22]. For a more detailed insight into hybrid systems, the reader is referred to [23].

The problem is solved as an hybrid optimal control problem in which, given the dynamic model of the aircraft (continuos dynamic), the initial and final states, a set of path constraints, and a set of waypoints (discrete dynamic), one has to find the control inputs, the switching times, the optimal sequence of waypoints and the corresponding trajectory of the aircraft that minimize a certain objective function during the flight, e.g., minimize fuel cost and overfly costs.

The complete layout of waypoints in the European airspace is reduced and the waypoints are gathered into a small number of clusters. The aircraft is constrained to pass through one waypoint inside every cluster of waypoints. The presence of the point constraints together with the fact that the sequence of waypoints is undefined makes the optimal control problem particularly difficult to solve. The hybrid optimal control problem is converted into a MINLP problem first making the unknown switching times part of the state, then introducing binary variable to enforce the constraint of passing once through each cluster of waypoints and finally applying a collocation method based on Gauss-Lobatto quadrature rules [24], [25] to convert dynamic equation of the system into constraints. The resulting MINLP problem has been solved using a branch and bound algorithm, Bonmin [26]. Three cases are studied and the numerical results are reported.

The paper is organized as follows: first, in Section II, we present aircraft dynamics, wind dynamics, waypoint data and overfly costs. The hybrid optimal control problem is stated in Section III, and it is reformulated in Section IV as a MINLP problem. The approach to its resolution is described in Section V and in Section VI numerical results are reported. Finally, Section VII contains the conclusions and a description of future work.

II. Aircraft dynamics

For finding aircraft trajectory purposes, it is commonly assumed to consider a 3 Degree Of Freedom (DOF) dynamic model that describes the point variable-mass motion of the aircraft over a flat earth model.

A 3D flight plan can be subdivided into a sequence of flight phases that can be regarded as symmetric flights either in a vertical or horizontal plane. The hypothesis of symmetric flight in a vertical or a horizontal plane allows the dynamic equation of motion of the aircraft to be simplified. Since this paper focuses on cruise phase, the equations for Symmetric flight in horizontal plane are presented.

A. Horizontal flight dynamics

The vertical component of wind, V_{wz_h}, and the first derivatives, $\dot{V}_{wx_h}$ and $\dot{V}_{wy_h}$, are not considered due to its low influence. We consider a spherical earth model. $\frac{2h}{R_e} << 1$, so the influence of all terms regarding normal acceleration is dismissed. Wind terms are expressed in terms of the fixed reference and projected into the reference attached to the aircraft. A standard atmosphere is defined with $\Delta_{ISA} = 0$, and a parabolic drag polar for C_D is assumed, i.e, $C_D = C_{D0} + KC_l^2$.. The airplane is a conventional jet airplane and BADA 3.6[1] is used as aircraft performance models.

The 3DOF equations governing the translational Horizontal 2D motion of an airplane are the following:

$$\begin{aligned}
m\dot{V} &= T - D, \\
mV\dot{\chi}\cos\gamma &= L\sin\mu, \\
L\cos\mu &= mg, \\
\dot{\lambda} &= \frac{V\cos\chi + V_{wx_h}}{R_e\cos\theta}, \\
\dot{\theta} &= \frac{V\sin\chi + V_{wy_h}}{R_e}, \\
\dot{m} &= -\eta T,
\end{aligned} \tag{1}$$

In general, the engine thrust T and bank angle μ are the control variables of the aircraft, that is $u = (T, \mu)$. The thrust is commanded by the engine throttle and the bank angle is commanded combining rudder and ailerons trims. The state vector, x, will be: $x = (\lambda, \theta, V, \chi, m)$, where λ is de longitude, θ the latitude, V the True Air Speed, χ the heading angle and m the mass of the aircraft.

The path constraints of the problem are those that define aircraft's flight envelope and can also be consulted in BADA database manual [27].

[1]http://www.eurocontrol.int

$$
\begin{aligned}
&& 0 \le CL \le CL_{max}, \\
0 \le h \le \min[h_{M0}, h_u], && T \le T_{max}, \\
C_{V_{min}} V_{stall} \le V \le V_{Mo}, && \mu \le \mu_{max,civ}, \\
M \le M_{M0}, && \dot{V} \le a_{l,max(civ)}, \\
m_{min} \le m \le m_{max}, && \dot{\gamma} \le \frac{a_{n,max(civ)}}{V},
\end{aligned}
\tag{2}
$$

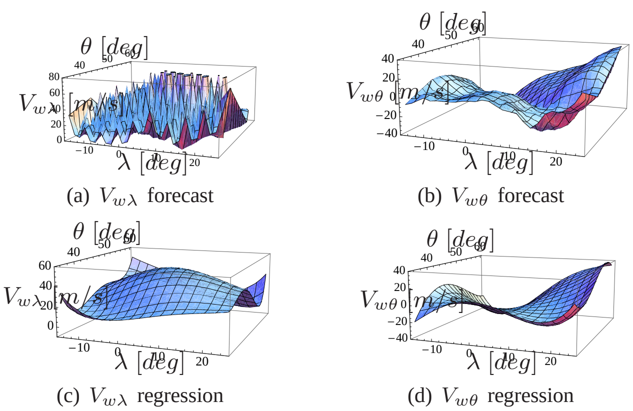
where $h_u = h_{max} + G_t(\Delta T_{ISA} - C_{Tc,4}) + G_W(m_{max} - m)$, and $C_{V_{min}} = 1.2$.

B. Wind data

In order to include wind dynamics in the aircraft dynamical model (1) and solve the resulting MIOCP, an analytical function for wind dynamics is needed.

Therefore, we represent the wind function as a nonlinear regression of GRIB data provided by the National Oceanic and Atmospheric Administration (NOAA [2]) wind forecasts. GRIB data are given in spherical coordinates, i.e., longitude, latitude and altitude. In particular, we adjust these GRIB tabular data to analytical functions, so that both west-east component, $V_{w\lambda} = f(\lambda, \theta)$, and south-noth component, $V_{w\theta} = f(\lambda, \theta)$, are functions of λ and θ. As the component perpendicular to earth, V_{wh}, is negligible, we have set it to zero.

The wind forecast of october the 20th, 2010 in the European region has been considered. We convert those tabular data into analytical functions by means of nonlinear regression. A 4th degree polynomial is fitted to the data. The goodness of fit, measured in terms of R-Squared parameter, yielded 0.76 for $V_{w\lambda}$ and 0.88 for $V_{w\theta}$. Real data and analytical functions are presented in Figure 1 at 250 [Hpa] (h=10393 [m]).



(a) $V_{w\lambda}$ forecast (b) $V_{w\theta}$ forecast

(c) $V_{w\lambda}$ regression (d) $V_{w\theta}$ regression

Fig. 1. Wind data at 250 [HPa] -h=10395 [m]-

C. Waypoints data

As defined above, a flight plan must be defined specifying a certain number of waypoints through which the aircrafts is going to fly. AUGUR [3], a tool developed by Eurocontrol, provides a list of current en-route waypoints and navaids in the ECAC (European Civil Aviation Conference) area which can be selected by ICAO (International Civil Aviation Organization) identifier.

[2]http://www.noaa.gov/

[3]http://augur.ecacnav.com

Fig. 2. European waypoints: Airport, Navaid and en-route waypoints of Portugal (blue), Spain (red), France (green), Italy (pink), England (violet), Germany (blue), Switzerland (yellow) and Benelux (orange)

Figure 2 illustrates all Airport, Navaid and en-route waypoints in western Europe. It gives a qualitative measure of the complexity of defining an efficient flight plan.

D. En-Route Overfly Charges

The en-route charge of a flight shall be calculated in accordance with $R = UR \times N$, in which R is the charge, UR is the unit rate and $N = d \times p$ is the number of service units corresponding to each flight, where d is the flight distance factor accomplished, and p the weight factor of the aircraft.

The distance factor shall be obtained obtained by diving by 100 the number of kilometers in the great circle distance between: the aerodrome of departure within, or the entry point into, the defined airspace, and; the aerodrome of first destination within, of the exit point of that airspace. These entry and exit points are those where the route described in the flight plan crosses the lateral limits of the different FIRs. The weight factor shall be $p = \sqrt{MTOW[ton]/50}$.

The basic unit rates, from January 1st 2010, for some european countries are: Spain, 84.1 €; France, 65.1 €; Germany, 68.99 €; Italy, 68.64 €; Switzerland, 75.05 €(1.51 CHF = 1 €exchange rate); Belgium and Luxemburg, 76.59 €; Netherlands, 65.8 €.

III. Description of the Problem and formulation

We study the problem of finding the control inputs $T(t)$ and $\mu(t)$ that steer the state of an aircraft whose dynamic model is given by (1) from the initial state x_I to the final state x_F, passing through a set of waypoints and minimizing the direct operating cost (in this case, fuel consumption and overfly costs) during the horizontal flight.

Let wp_I and wp_F be the aircraft positions that correspond to x_I and x_F, respectively. We define a number of clusters, n_{cl}, each cluster containing some determined number of waypoints, n_{wp}, so that the aircraft must pass through one waypoint out of n_{wp} inside cluster i, $i = 1 \ldots n_{cl}$.

Let $P = \{wp_{1,1}, \ldots, wp_{n_{cl},n_{wp}}\}$ be the set of possible waypoints that correspond to the set of possible states $\{x_{1,1}, \ldots, x_{n_{cl},n_{wp}}\}$. Thus, we can say that the aircraft is constrained to pass through n_{cl} waypoints of P when moving between wp_I and wp_F. The order in which the waypoints of P are overflown is specified by the cluster index, i.e., $i = 1 \ldots n_{cl}$, but the waypoints inside every cluster, i.e., $wp_{i,j},\ j \ldots n_{wp}$ must be determined.

If n_{cl} is the number of clusters of the set P, $n_f = n_{cl} + 1$ phases can be identified during the motion of the aircraft. More precisely, a phase starts when the aircraft position coincides with wp_I or when it passes through one of the points of the set P and ends when the aircraft passes through another point of the set P or when it reaches the final point wp_F. Let

$$t_I = \tilde{t}_0 \leq \tilde{t}_1 \leq \cdots \leq \tilde{t}_{n_{cl}} \leq \tilde{t}_{n_f} = t_F$$

be the switching times between phases. Thus for $t \in [\tilde{t}_k, \tilde{t}_{k+1}]$ the system is in phase k, $k = 0, \ldots, n_f - 1$. The duration of each phase must also be determined.

Let $\delta(t)$ the binary control function that indicates which point of the set P is visited by the aircraft at time $\tilde{t}_k$, i.e. at the beginning of phase k. It is a vector function whose components are piecewise constant with jumps at times $\tilde{t}_k, k = 1, \ldots, n_f - 1$, with a single nonzero component in each phase. The function $\delta(t)$ during phase k will be denoted by $\delta_k(t)$. Component i, j of $\delta_{k_{i,j}}(t_k)$, $\delta_{k_{i,j}}(t_k) = 1$ means that the aircraft flies through waypoint $wp_{i,j}$ at time $\tilde{t}_k$. Abusing notation, $\delta_{k_{i,j}}(t_k)$ will be simply denoted by $\delta_{i,j}$. Let $x_k(t)$ be the state variables of the optimal control problem during phase k. Thus, the point constraints can be expressed as follows

$$x_k(\tilde{t}_k) = \sum_{j=1}^{n_{wp}} \delta_{i,j}(\tilde{t}_k)\ x_{i,j}, k = i = 1, \ldots, n_f - 1 \qquad (3)$$

which means that, if $\delta_{i,j}(\tilde{t}_k) = 1$ at the beginning of phase k, it will be $x_k(\tilde{t}_k) = x_{i,j}$, that is, the aircraft will be in configuration $x_{i,j}$ and the associated waypoint position $wp_{i,j}$. Additional constraints are

$$\sum_{j=1}^{n_{wp}} \delta_{i,j}(\tilde{t}_k) = 1,\ i = 1 \ldots n_{cl} \qquad (4)$$

$$\delta_{i,j}(\tilde{t}_k) \leq 1,\ i = 1 \ldots n_{cl}. \qquad (5)$$

Condition (4) means that the aircraft must pass only through a single point, $wp_{i,j}$, in cluster i at time $\tilde{t}_k$, whereas condition (5) means that the aircraft may pass or not through every one of the waypoints $wp_{i,j}$.

The objective functional to be minimized is

$$J = \sum_{k=0}^{n_f-1} \int_{\tilde{t}_k}^{\tilde{t}_{k+1}} \dot{m}_k(t)\ dt\ + \sum_{i=1}^{n_{cl}} \sum_{j=1}^{n_{wp}} \delta_{i,j} OC_{wp_{i,j}}, \qquad (6)$$

where $\dot{m}_k(t)$ is the fuel flow of the aircraft during phase k and $OC_{wp_{i,j}}$ is the Overfly Cost (OC) associated to each of the waypoints. This cost functional represents a measure of the direct operating cost during the flight of an aircraft, which among others counts with the cost of the burnt fuel and the navigation fees due to overflying certain regions of the airspace.

1) General formulation of the optimal control problem: The aircraft flight planning problem stated above is a particular case of a multiphase mixed integer optimal control problem which can be stated in a more general form as follows [28, Chapter 1], [29]

$$\min \sum_{k=0}^{n_f-1} [\int_{\tilde{t}_k}^{\tilde{t}_{k+1}} L_k[x_k(t), u_k(t), \delta_k(t), v, z, t]dt + E_k[x_k(\tilde{t}_{k+1}), v, z]\ ] \qquad (7)$$

subject to

$$x_k'(t) = f_k[x_k(t), u_k(t), \delta_k(t), v, z, t], \quad t \in [\tilde{t}_k, \tilde{t}_{k+1}], \qquad (8)$$

$$g_k[x_k(t), u_k(t), \delta_k(t), v, z, t] = 0, t \in [\tilde{t}_k, \tilde{t}_{k+1}], \qquad (9)$$

$$c_k[x_k(t), u_k(t), \delta_k(t), v, z, t] \leq 0, t \in [\tilde{t}_k, \tilde{t}_{k+1}], \qquad (10)$$

$$r^{\text{ineq}}[x_{k_0}(t_0), x_{k_1}(t_1), \ldots, x_{k_{n_r}}(t_{n_r}), v, z] \leq 0, \qquad (11)$$

$$r^{\text{eq}}[x_{k_0}(t_0), x_{k_1}(t_1), \ldots, x_{k_{n_r}}(t_{n_r}), v, z] = 0, \qquad (12)$$

$$x_{k+1}(\tilde{t}_{k+1}) = tr[x_k(\tilde{t}_{k+1}), u_k(\tilde{t}_{k+1}), v, z], \qquad (13)$$

for $k = 0, \ldots, n_f - 1$.

a) Continuous variables: $t \in [t_I, t_F] \subset \mathbb{R}$ is the time, $x_k(t) \in \mathbb{R}^{n_{x_k}}$ is the state variable in phase k whose time derivative is $x_k{}'(t) \in \mathbb{R}^{n_{x_k}}$ and $u_k(t) \in \mathbb{R}^{n_{u_k}}$ is the control function in phase k, which is assumed to be measurable. Variable $z \in \mathbb{R}^{n_z}$ represents a vector of parameters.

b) Discrete (integer and binary) variables: Let $\delta(t)$: $[t_I, t_F] \mapsto \mathbb{R}^{n_\delta}$ be a measurable function. A time dependent variable $\delta(t)$ is called an integer variable or an integer control function if it takes values in $\mathbb{Z}^{n_\delta}$. If takes values in $\{0, 1\}^{n_\delta}$ it is called a binary variable or binary control function. Let $v \in \mathbb{R}^{n_v}$ a vector. A time independent variable v is called integer variable if takes values in $\mathbb{Z}^{n_v}$, and binary variable if takes values in $\{0, 1\}^{n_v}$. We assume that integer variables can take values in a finite set. We suppose that $\delta(t)$ is piecewise constant in $[t_I, t_F]$ with jumps only at times $t \in \{\tilde{t}_1, \ldots, \tilde{t}_{n_f-1}\}$ which are the instants when a discontinuity in the binary control function $\delta(t)$ may occur. If at time there is a discontinuity in at least one of its components we say that a switching occurred and this time is called the switching time. $\delta_k(t) \in \mathbb{Z}^{n_\delta}$ is the value of variable $\delta(t)$ in phase k.

c) Objective functional: The terms of the objective functional (7) are in Bolza form and contains a Lagrange term $\int_{t_k}^{t_{k+1}} L_k[x_k(t), u_k(t), \delta_k(t), v, z, t]dt$ and a Mayer term $E_k[x_k(t_{k+1}), v, z]$. Both L and E are assumed to be twice differentiable.

d) Constraints: Equation (8) and Equation (9) with $f_k \in \mathbb{R}^{n_{f_k}}$ and $g_k \in \mathbb{R}^{n_{g_k}}$ are the equations of the differential-algebraic model of the system in phase k.

Equations (10) with $c_k \in \mathbb{R}^{n_{c_k}}$ are the path constraints in phase k.

Equations (11) with $r^{ineq} \in \mathbb{R}^{n_{r^{ineq}}}$ and Equations (12) with $r^{eq} \in \mathbb{R}^{n_{r^{eq}}}$ are the inequality and equality multi point constraints, respectively, which are assumed to be twice differentiable. In these equations k_i denotes the index of the phase that contains t_i, that is, $t_i \in [\tilde{t}_{k_i}, \tilde{t}_{k_{i+1}}]$. In our case the number of interior point constraints n_r coincides with the number of clusters n_{cl}.

The dimensions $n_{x_k}, n_{u_k}, n_\delta, n_{c_k}, n_{r^{ineq}}, n_{r^{eq}}, n_{f_k}, n_{g_k}$ are not necessarily identical for each phase.

Equations (13) are the transition conditions between phases which are usually of the form $x_{k+1}(\tilde{t}_{k+1}) = x_k(\tilde{t}_{k+1})$.

The solution of this problem is given by the set $\{x(t), u(t), \delta(t), v, z | t \in [t_I, t_F]\}$.

e) Hybrid dynamical systems: The set of dynamic systems (8) is called hybrid dynamical system and the problem (7)-(13) is actually a hybrid optimal control problem formulated as a multiphase Mixed Integer Optimal Control Problem (MIOCP).

The binary control function $\delta(t)$ in $[t_I, t_F]$ defines both the untimed sequence of phases $(k_0, k_1, \ldots, k_{n_f-1})$ and the sequence of switching times $(\tilde{t}_1, \ldots, \tilde{t}_{n_f-1})$. Switches can be either autonomous or controlled. For instance, autonomous switches may occur when the system reaches a prescribed set of the state space expressed by Equations (13). On the contrary, controlled switches take place in response to control inputs which in our case are determined by the solution of the optimal control problem (7)-(13).

In this work we assume that the number of phases is known, both the untimed sequence of phases and the sequence of switching times are not known, and the dynamic equation of the system does not change across different phases, i.e., the aircraft flies in all phases with clean configuration and in the horizontal plane.

f) Relaxed optimal control problem: In the following sections the concept of relaxed optimal control problem will be used. The relaxation of a MIOCP is the optimal control problem obtained replacing integrality assumption for $\delta_k(t)$ and v with the conditions $\delta_k(t) \in [0,1]^{n_{\delta_k}}$ and $v \in [0,1]^{n_v}$, respectively.

IV. Problem Reformulation

A. Incorporating Switching Times

For the sake of simplicity of exposition of the method consider the following simplified version of the optimal control problem (7)-(13)

$$\min \sum_{k=0}^{n_f-1} \left[\int_{\tilde{t}_k}^{\tilde{t}_{k+1}} L_k[x(t), u(t), t] dt + E_k \left[x\left(\tilde{t}_{k+1}\right) \right] \right] \quad (14)$$

subject to

$$x'(t) = f_k[x(t), u(t), t]. \quad (15)$$

where $\tilde{t}_0 \le \tilde{t}_1 \le \cdots \le \tilde{t}_{n_{cl}} \le \tilde{t}_{n_f}$ are the switching times between phases which must be determined.

This hybrid optimal control problem is converted into a conventional optimal control problem making the unknown switching times part of the state and introducing a new independent variable with respect to which the switching times are fixed [30], [31].

Without loss of generality, we can assume that $t_0 = \tilde{t}_0 = t_I = 0$ and $\tilde{t}_{n_f} = t_F = 1$.

Since the number of switches n_{cl} is known, we introduce the new state variables, $x_{n_x+1}, \ldots, x_{n_x+n_{cl}}$, such that $x_{n_x+k} = \tilde{t}_k$, with $x'_{n_x+k} = 0, k = 1, 2, \ldots, n_{cl}$. Let

$$\hat{x} = [x_1, \ldots, x_{n_x}, x_{n_x+1}, \ldots, x_{n_x+n_{cl}}]^T$$

be the extended state vector.

We introduce the new independent variable, s and choose an increasing sequence of n_{cl} values on the interval $[0,1]$ setting $s_k = k/(n_{cl}+1), k = 1, \ldots, n_{cl}$. However, any monotonically increasing sequence of n_{cl} values on the interval $[0,1]$ could be used for s_k.

We then establish a piecewise linear correspondence between time, t, and the new independent variable, s, so that for every chosen fixed point, s_k, $k = 1, \ldots, n_{cl}$, t equals $\tilde{t}_k$. The relation between s and t changes on each interval $\left[\tilde{t}_k, \tilde{t}_{k+1}\right]$.

As a result we obtain the following expressions for the change of variable

$$t = \begin{cases} (n_{cl}+1)\ x_{n_x+1}\ s, & 0 \le s \le \frac{1}{n_{cl}+1} \\ \ldots & \\ (n_{cl}+1)(x_{n_x+k+1} - x_{n_x+k})\ s + & \\ \quad (k+1)\ x_{n_x+k} - k\ x_{n_x+k+1}, & \frac{k}{n_{cl}+1} < s \le \frac{k+1}{n_{cl}+1} \\ \ldots & \\ (n_{cl}+1)(1 - x_{n_x+n_{cl}})\ s + & \\ \quad (n_{cl}+1)\ x_{n_x+n_{cl}} - n_{cl}, & \frac{n_{cl}}{n_{cl}+1} < s \le 1 \end{cases} \quad (16)$$

After introducing the new independent variable, in the interval $\frac{k}{n_{cl}+1} < s \le \frac{k+1}{n_{cl}+1}$ $k = 0, \ldots, n_{cl}$ the dynamic constraint (15) becomes

$$x'(s) = (n_{cl}+1)(x_{n+k+1} - x_{n+k}) \hat{f}_k[x(s), u(s), s], \quad (17)$$

where $\hat{f}_k[x(s), u(s), s] = f_k[x(t), u(t), t(s)]$, and

$$\hat{L}_k(\hat{x}(s), u(s), s) = (n_{cl}+1)(x_{n+k+1} - x_{n+k}) L_k[x(t), u(t), t(s)],$$

In this way the problem (14)-(15) can be rewritten as

$$\min \sum_{k=0}^{n_{cl}} \left[\int_{\frac{k}{n_{cl}+1}}^{\frac{k+1}{n_{cl}+1}} \hat{L}_k[\hat{x}(s), u(s), s] ds + E_k \left[\hat{x}\left(\frac{k+1}{n_{cl}+1} \right) \right] \right], \quad (18)$$

subject to

$$x'(s) = (n_{cl}+1)(x_{n+k+1} - x_{n+k}) \hat{f}_k[x(s), u(s), s]. \quad (19)$$

The new equivalent problem is a conventional optimal control problem. The last n_{cl} components of the optimal solution $\hat{x}^*$ of this problem will be the optimal switching times $\tilde{t}_k, k = 1, \ldots, n_{cl}$.

B. Optimal control resolution using collocation

In this section the numerical method employed to solve the optimal control problem (18)-(19) derived from the reformulation will be described. For the sake of simplicity of exposition consider the following simplified version of this problem

$$\min \int_{t_I}^{t_F} L[x(t), u(t), t]dt + E(t_F) \tag{20}$$

subject to

$$x'(t) = f[x(t), u(t), t]. \tag{21}$$

A collocation approach has been used in which integration rules are based on a particular family of Jacobi interpolating polynomials that give rise to the so called Gauss-Lobatto quadrature rules. In this class of numerical methods the optimal control problem is converted into a NonLinear Programming (NLP) problem.

The time interval $[t_I, t_F]$ is subdivided into n_d intervals whose endpoints are $\{\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_{n_d}\}$ with $\bar{t}_0 = t_I$ and $\bar{t}_{n_d} = t_F$ and a numerical integration scheme is used for the objective functional in (20) and Equation (21) in each subinterval $[\bar{t}_i, \bar{t}_{i+1}]$.

Consider a simplified form of differential equation (21), $dx/dt = f(t)$. Basic numerical integration methods to solve this differential equation rely on the trapezoidal rule

$$x(\bar{t}_{i+1}) - x(\bar{t}_i) = \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(t)dt \approx \frac{h_i}{2}[f(\bar{t}_i) + f(\bar{t}_{i+1})],$$

with $h_i = (\bar{t}_{i+1} - \bar{t}_i)$, where the integrand is approximated with a linear function, and on the Simpson's rule

$$x(\bar{t}_{i+1}) - x(\bar{t}_i) = \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(t)dt \approx \frac{h_i}{6}[f(\bar{t}_i) + 4f(\bar{t}_{i,C}) + f(\bar{t}_{i+1})], \tag{22}$$

in which the integrand is approximated using a quadratic polynomial which depends on the values of the integrand at the endpoints of the interval $[\bar{t}_i, \bar{t}_{i+1}]$ and at the midpoint $\bar{t}_{i,C} = (\bar{t}_{i+1} - \bar{t}_i)/2$ of this interval. These points are called collocation points. Both the trapezoid rule and the Simpson's rule belong to the so called Gauss-Lobatto family of integration rules in which the degree of the integrated polynomial coincides with the number of discrete value of the integrand used to generate the interpolating polynomial. Thus, the trapezoid rule is the second-degree rule and Simpson's rule is the third-degree Gauss-Lobatto integration rule.

Both the trapezoid rule and the Simpson's rule can be used to derive iterative schemes to numerically solve differential equations of the form $dx/dt = f(x)$ which can be expressed in the form of constraints. If $x_i = x(\bar{t}_i)$ and $x_{i+1} = x(\bar{t}_{i+1})$, from the trapezoid rule we obtain the constraint

$$C_T(x_i, x_{i+1}) = x_i - x_{i+1} + \frac{h_i}{2}[f(x_i) + f(x_{i+1})] = 0,$$

called the trapezoid system constraint, and from the Simpson's rule the constraint

$$C_S(x_i, x_{i+1}) = x_i - x_{i+1} + \frac{h_i}{6}[f(x_i) + 4f(x_{i,C}) + f(x_{i+1})] = 0 \tag{23}$$

called the Hermite-Simpson system constraint in which the approximation of $x(t)$ at $\bar{t}_{i,C}$ is given by

$$x_{i,C} = \frac{x_i + x_{i+1}}{2} + \frac{h_i}{8}[f(x_i) - f(x_{i+1})]. \tag{24}$$

Constraint (23) is obtained imposing $f(x_{i,C}) = x'_{i,C}$.

For numerical calculations, we will use a Hermite-Simson collocation method [24].

V. MINLP Solution Approach

Applying all the transformations described in the previous sections, we obtain a MINLP problem, whose form is omitted for the sake of brevity.

In this problem, fixing the variables $\delta_{i,j}$ $i = 1, \ldots, n_{cl}$, $j = 1, \ldots, n_{wp}$ is equivalent to fixing the sequence of waypoints and if this is done the MINLP problem becomes a regular flight planning problem. A simple algorithmic approach could therefore be to enumerate all possible values for $\delta_{i,j}$, solve the associated motion planning problems and pick the best one. A rapid calculation of the number of problems if one follows this approach shows that it is impractical for more than a handful of waypoints. A common approach to try to address bigger problems is to do an implicit enumeration via the branch-and-bound algorithm [32], [33]. Branch and bound has now become a standard algorithm for integer programming (see for example [34] and reference therein). We give below a brief sketch of it in our context, in particular we try to stress out the particularities that arise in trying to solve the MINLP problem.

Branch and bound is a divide-and-conquer method. The dividing (branching) is done by partitioning the set of feasible solutions into smaller and smaller subsets. The conquering (fathoming) is done by bounding the value of the best feasible solution in the subset and discarding the subset if its bound indicates that it cannot contain an optimal solution. Branch and bound is an exact algorithm when the bound used in the fathoming phase is a valid lower bound. Our case is particular in that, obtaining a valid lower bound of the MINLP problem is usually a daunting task. Therefore, we don't use a true lower bound but just a lower approximation of the MINLP problem. In that case the procedure is heuristic (i.e. does not return the exact optimal solution). Of course, the quality of the final solution found depends on the quality of the approximation of the bound. We don't have a theoretical guarantee on the quality of the approximation but we will try to assess it in the computational section of this paper.

The major question to apply the branch-and-bound algorithm is which method to use to bound or approximate the value of the problem in a subset. In our case, we use NonLinear Programming (NLP). This variant of branch-and-bound is usually naturally called NLP based branch-and-bound or NLP BB for short (for more details, see for example [35] and references therein).

Several solvers such as MINLP-BB [36] and SBB [37] implement the NLP BB algorithm. In our case, we used the solver Bonmin [26]. Bonmin is an open-source MINLP solver implementing several different algorithms for solving mixed integer nonlinear optimization problems. Source code and binaries of Bonmin are available from COIN-OR (http://www.coin-or.org). Bonmin may be called from both the AMPL and GAMS modeling languages or be used via a web interface on NEOS (http://www-neos.mcs.anl.gov).

VI. Numerical Experiments

We consider the horizontal motion of a commercial aircraft constrained to pass through a certain number of waypoints. We analyze three cases, namely Case A, Case B and Case C. In these experiments the underlying idea is to proof the consistency and robustness of this formulation applied to the cruise phase optimization. We solve realistic problems taking into consideration real waypoints (Section II-C), wind conditions (Section II-B) and overfly costs (Section II-D). In Case A we neither consider wind nor overfly costs; in Case B we consider wind dynamics; and in Case C we consider overfly costs and neglect wind effects.

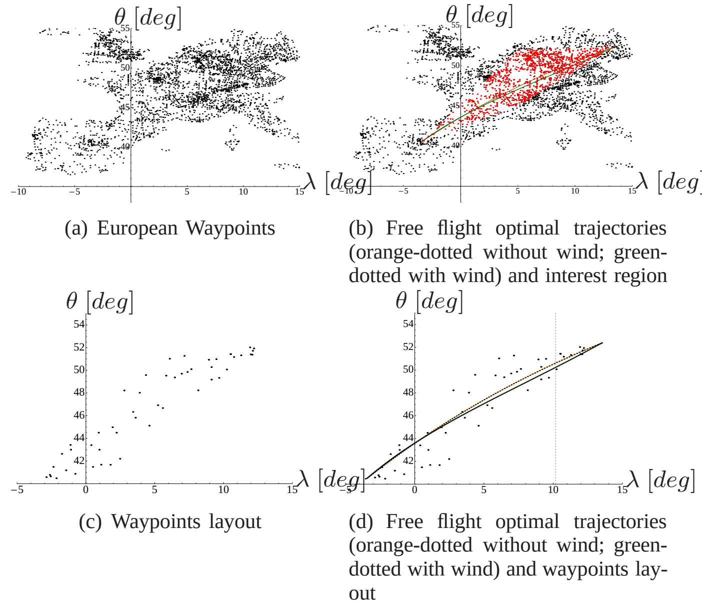
In all cases, an Airbus A-320 performs the cruise phase of a flight Madrid-Berlin with the following boundary conditions: $\lambda_{t_I} = -3.56$ [deg], $\theta_{t_I} = 40.47$ [deg], $h_{t_I} = 10393$ [m], $V_{t_I} = 220$ [m/s], $\gamma_{t_I} = 0$ [deg], $\chi = 0$ [deg], $\mu = 0$ [deg], $m_{t_I} = 64000$ [Kg]; $\lambda_{t_F} = 13.52$ [deg], $\theta_{t_F} = 52.38$ [deg], $h_{t_F} = 10393$ [m].

Since the big amount of waypoints makes it impossible an efficient computation, it is mandatory a reduction of their number through a proper selection of potential candidates, denoted $wp_{i,j},\ i = 1 \dots n_{cl},\ j = 1 \dots n_{wp}$, where n_{cl} is the number of clusters, and n_{wp} the number of waypoints per cluster.

The parameter n_{cl} will be given by the minimum number of waypoints to be flown trough when defining the flight plan; n_{wp} is a trade off between accuracy in the optimal solution and the need for efficiently computing trajectories at strategic level. We choose, for the three cases, $n_{cl} = 10$ clusters of $n_{wp} = 5$ waypoints each, so that the aircraft must fly through 1 out of $n_{wp} = 5$ waypoints inside each cluster. The continuous motion of the aircraft is given by Equation 1, while the discrete dynamics are given by the sequence of waypoints.

For clustering, we start considering all waypoints in Figure 2, retaining only the en-route waypoints and eliminating all navaid and airport waypoints, see Figure 3(a). Then, we generate the free-flight optimal trajectories (with and without wind), see Figure 3(b). Afterwords, we define an interest region close to the free-flight optimal trajectories, see figure 3(b). Finally, we take $n_{cl} = 10$ clusters homogeneously distributed along the free-flight optimal trajectory. From each cluster, we randomly choose $n_{wp} = 5$ waypoints.

The waypoints layout, and its coordinates together with the associated costs are shown in Figure 3(c) and in Table I, respectively. Notice that the associated waypoint cost, $C_{wp_{i,j}}$, has been assigned considering the basic unit rates of the different countries (see Section II-D). To illustrate how the cost of the different flight regions affects the optimal sequence of waypoints, the assigned costs does not necessary correspond to the real cost of the country in which the waypoint lays.



(a) European Waypoints

(b) Free flight optimal trajectories (orange-dotted without wind; green-dotted with wind) and interest region

(c) Waypoints layout

(d) Free flight optimal trajectories (orange-dotted without wind; green-dotted with wind) and waypoints layout

Fig. 3. Free flight optimal trajectories and waypoints layout

$wp_{1,1} = (40.58, -2.88)$; 84.11 €	$wp_{1,2} = (40.78, -2.63)$; 84.11 €
$wp_{1,3} = (40.65, -2.57)$; 84.11 €	$wp_{1,4} = (41.48, -2.37)$; 84.11 €
$wp_{1,5} = (40.48, -2.15)$; 65.1 €	$wp_{2,1} = (42.63, -1.75)$; 65.1€
$wp_{2,2} = (41.18, -1.45)$; 65.1€	$wp_{2,3} = (43.42, -1.13)$; 65.1€
$wp_{2,4} = (43, -1.10)$; 68.64 €	$wp_{2,5} = (40.87, -0.78)$; 68.64 €
$wp_{3,1} = (43.40, 0.38)$; 65.1 €	$wp_{3,2} = (41.48, 0.5)$; 65.1 €
$wp_{3,3} = (44.5, 0.92)$; 75.05 €	$wp_{3,4} = (43, 0.97)$; 68.64 €
$wp_{3,5} = (41.68, 1.1)$; 65.1 €	$wp_{4,1} = (41.67, 1.78)$; 65.1 €
$wp_{4,2} = (44.98, 1.93)$; 65.1 €	$wp_{4,3} = (44.50, 2.22)$; 65.1 €
$wp_{4,4} = (42.20, 2.47)$; 65.1 €	$wp_{4,5} = (48.22, 2.78)$; 65.1 €
$wp_{5,1} = (46.33, 3.42)$; 68.99 €	$wp_{5,2} = (45.82, 3.60)$; 65.1 €
$wp_{5,3} = (48.02, 3.90)$; 65.1 €	$wp_{5,4} = (49.58, 4.35)$; 65.1 €
$wp_{5,5} = (45.12, 4.60)$; 76.59 €	$wp_{6,1} = (46.92, 5.23)$; 68.99 €
$wp_{6,2} = (46.67, 5.58)$; 65.80 €	$wp_{6,3} = (49.53, 5.82)$; 65.80 €
$wp_{6,4} = (51.02, 6.08)$; 68.99 €	$wp_{6,5} = (49.37, 6.45)$; 76.59 €
$wp_{7,1} = (49.70, 6.95)$; 76.59 €	$wp_{7,2} = (51.27, 7.15)$; 65.80 €
$wp_{7,3} = (49.85, 7.37)$; 65.80 €	$wp_{7,4} = (50.10, 7.65)$; 65.80 €
$wp_{7,5} = (48.23, 8.15)$; 65.80 €	$wp_{8,1} = (50.93, 8.92)$; 68.99 €
$wp_{8,2} = (50.28, 9.12)$; 68.99 €	$wp_{8,3} = (49.18, 9.12)$; 68.99 €
$wp_{8,4} = (50.98, 9.47)$; 68.99 €	$wp_{8,5} = (49.33, 9.67)$; 68.99 €
$wp_{9,1} = (50.07, 10.23)$; 68.99 €	$wp_{9,2} = (51.43, 10.50)$; 68.99 €
$wp_{9,3} = (51.42, 10.52)$; 68.99 €	$wp_{9,4} = (51.17, 10.75)$; 68.99 €
$wp_{9,5} = (51.32, 11.27)$; 68.99 €	$wp_{10,1} = (52.02, 11.92)$; 68.99 €
$wp_{10,2} = (51.40, 11.97)$; 68.99 €	$wp_{10,3} = (51.38, 12.08)$; 68.99 €
$wp_{10,4} = (51.72, 12.12)$; 68.99 €	$wp_{10,5} = (51.93, 12.20)$; 68.99 €

TABLE I

Cordinates of the Waypoints, $wp_{i,j} = (\theta[deg], \lambda[deg])$, and associated cost, $C_{wp_{i,j}}$.

1) Case A: The total consumed fuel in this experiment is 5259.35 [Kg]. The optimal sequence of waypoints, denoted by the active set of binary variables $\delta_{i,j}$, with $i = 1 \dots n_{cl}, j = 1 \dots n_{wp}$, is given in Table II. The optimal trajectory is given in Figure 4 (orange-dotted line). The switching times between phases and total flight time are listed in Table V. The state and control variables of the optimal solution are shown in Figure 5 (orange-dotted line). The computation time to find the solution was 2442.3 [s].

2) Case B: The total consumed fuel in this experiment is 4839.34 [Kg]. The optimal sequence of waypoints, denoted by the active set of binary variables $\delta_{i,j}$, with $i = 1 \dots n_{cl}, j = 1 \dots n_{wp}$, is given in Table III. The optimal trajectory is given in Figure 4 (green-dotted line). The switching times between phases and total flight time are listed in Table V. The state and control variables of the optimal solution are shown in Figure 5 (green-dotted line). The computation time to find the solution was 6820.9 [s].

3) Case C: To take overfly cost into consideration, we use Equation (6) as objective function, considering that $OC_{wp_{ij}} = UR \times d \times p$, where UR is modeled as a cost due to overflying a determined waypoint, $C_{wp_{i,j}}$, listed in Table I, d is modeled as a constant consisting on dividing the total distance to be flown (1854.98 [km]) by 100 and by the number of waypoints to overfly, $n_{cl} = 10$, and p is defined as in Section II-D. We also assume that 1 [kg] of fuel burnt costs 1 €.

The total consumed fuel in this experiment is 5265.35 [Kg]. The optimal sequence of waypoints, denoted by the active set of binary variables $\delta_{i,j}$, with $i = 1 \dots n_{cl}, j = 1 \dots n_{wp}$, is given in Table IV. The optimal trajectory is given in Figure 4 (red-dotted line). The switching times between phases and total flight time are listed in Table V. The state and control variables of the optimal solution are shown in Figure 5 (red-dotted line). The computation time to find the solution was 6120.2 [s]. The overfly cost as defined above was 1587.65 €.

TABLE II
CASE A: SWITCHING SEQUENCE.

$\delta_{i,j}$	$\delta_{i,1}$	$\delta_{i,2}$	$\delta_{i,3}$	$\delta_{i,4}$	$\delta_{i,5}$
$\delta_{1,j}$	0	0	0	1	0
$\delta_{2,j}$	0	0	0	1	0
$\delta_{3,j}$	0	0	1	0	0
$\delta_{4,j}$	0	1	0	0	0
$\delta_{5,j}$	1	0	0	0	0
$\delta_{6,j}$	0	0	0	0	1
$\delta_{7,j}$	0	0	1	0	0
$\delta_{8,j}$	0	0	0	1	0
$\delta_{9,j}$	0	0	1	0	0
$\delta_{10,j}$	0	0	0	0	1

TABLE III
CASE B: SWITCHING SEQUENCE.

$\delta_{i,j}$	$\delta_{i,1}$	$\delta_{i,2}$	$\delta_{i,3}$	$\delta_{i,4}$	$\delta_{i,5}$
$\delta_{1,j}$	0	0	0	1	0
$\delta_{2,j}$	0	0	0	1	0
$\delta_{3,j}$	0	0	1	0	0
$\delta_{4,j}$	0	1	0	0	0
$\delta_{5,j}$	0	1	0	0	0
$\delta_{6,j}$	1	0	0	0	0
$\delta_{7,j}$	0	0	0	0	1
$\delta_{8,j}$	0	0	1	0	0
$\delta_{9,j}$	1	0	0	0	0
$\delta_{10,j}$	0	0	1	0	0

A. Discussion on the experiments

In Case A and Case B, we can observe that we achieve consumptions slightly far from the free-flight optimal consumptions without and with wind, respectively. Indeed, the existing consumption gap between waypoints constrained trajectories (Figure 4) and (without and with wind) free flight trajectories (Figure 3(d)) is a good caliber of how good is the layout of selected waypoints, i.e., the closer we are to the optimal solution the better waypoints layout have been selected. The above mentioned gap is, respectively, 53.57[kg] and 39.05[kg] for Case A and Case B (Notice that Case A is compared with the without wind free flight optimal consumption and Case B is compared with the with wind one. See Table VI for more details).

Notice that in Case C the assigned costs are significative enough to generate a trajectory substantially different from

TABLE IV
CASE C: SWITCHING SEQUENCE.

$\delta_{i,j}$	$\delta_{i,1}$	$\delta_{i,2}$	$\delta_{i,3}$	$\delta_{i,4}$	$\delta_{i,5}$
$\delta_{1,j}$	0	0	0	1	0
$\delta_{2,j}$	0	0	0	1	0
$\delta_{3,j}$	0	0	1	0	0
$\delta_{4,j}$	0	1	0	0	0
$\delta_{5,j}$	0	1	0	0	0
$\delta_{6,j}$	0	1	0	0	0
$\delta_{7,j}$	0	0	0	0	1
$\delta_{8,j}$	0	0	1	0	0
$\delta_{9,j}$	1	0	0	0	0
$\delta_{10,j}$	0	0	1	0	0

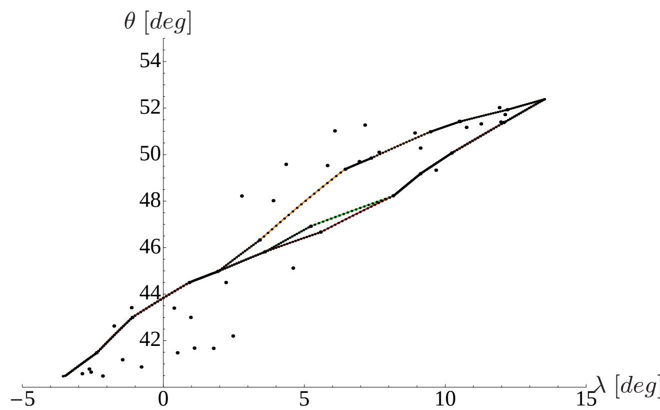
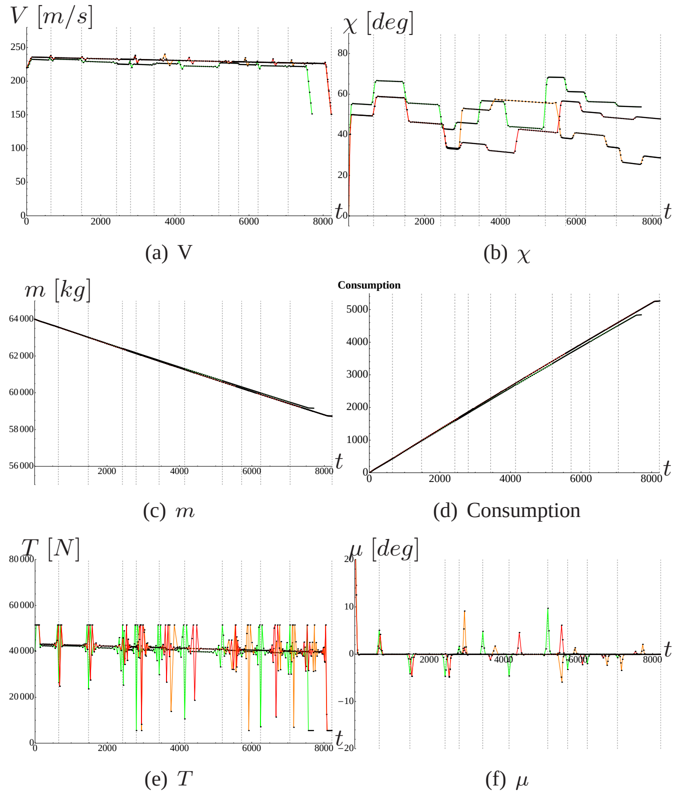


Fig. 4. Optimal trajectory and optimal sequence of waypoints: Case A, orange-dotted line; Case B, green-dotted line; Case C, red-dotted line.

Case A [s]	Case B [s]	Case C [s]
$t_1 = 681.52$	$t_1 = 655.87$	$t_1 = 681.52$
$t_2 = 1531.79$	$t_2 = 1480.77$	$t_2 = 1531.79$
$t_3 = 2533.19$	$t_3 = 2425.09$	$t_3 = 2533.19$
$t_4 = 2956.31$	$t_4 = 2803.01$	$t_4 = 2946.43$
$t_5 = 3765.97$	$t_5 = 3437.77$	$t_5 = 3641.18$
$t_6 = 5544.11$	$t_6 = 4143.38$	$t_6 = 4418.89$
$t_7 = 5916.23$	$t_7 = 5184.43$	$t_7 = 5554.99$
$t_8 = 6774.41$	$t_8 = 5717.58$	$t_8 = 6113.63$
$t_9 = 7162.08$	$t_9 = 6244.45$	$t_9 = 6672.24$
$t_{10} = 7733.49$	$t_{10} = 7055.88$	$t_{10} = 7536.55$
$t_f = 8214.78$	$t_f = 7696.14$	$t_f = 8219.84$

TABLE V
SWITCHING TIMES AND TOTAL FLIGHT TIME



(a) V (b) χ (c) m (d) Consumption (e) T (f) μ

Fig. 5. State and control variables: Case A, orange-dotted line; Case B, green-dotted line; Case C, red-dotted line.

TABLE VI

Case	Consumption [kg	Gap [kg]	OC [€]
Free flight without wind	5205.68	-	-
Free-flight with wind	4800.29	-	-
Case A	5259.35	53.57	-
Case B	4839.34	39.05	-
Case C	5265.35	59.57	1587.65

Case A.

These experiments illustrate how the optimal trajectory changes when we increase the accuracy of the model, introducing for instance wind effects or overfly costs. Thus, our MINLP approach provides a very powerful tool to tackle such problems.

VII. Conclusions and future work

Since free-flight does not seem to be implemented in the medium term, we have shown that combining an airspace structured in waypoints together with a more flexible continuous motion of the aircraft, e.g., not performing steady cruise, might be a very powerful tool for a more efficient strategic flight planning within a short and medium-term future ATM concept.

Our current research efforts are focused on improving the numerical efficiency of the method to reduce the computational time, a key factor to timely deliver flight plans, and, on getting closer to real flights, combining cruise with ascent and descent, to better model overfly costs, or to better reflect the airspace structure.

Acknowledgments

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