Analysis of the Effect of Uncertain Average Winds on Cruise Fuel Load

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***Abstract*—The required fuel load of an aircraft for a given cruise range with uncertain average wind (modelled by a uniform distribution) is studied in this work. The fuel mass probability density function is obtained using an approximate method previously developed by the authors. In addition, the Generalized Polynomial Chaos method is used to study the mean and typical deviation of the required fuel mass. The dynamics of mass evolution in cruise flight is defined by a nonlinear equation, which can be solved analytically to obtain the fuel mass; this exact solution is used to assess the accuracy of the proposed methods. Comparison of the numerical results with the exact analytical solutions shows an excellent agreement in all cases. The results are also compared with the Monte Carlo method, which requires a much larger computation time to obtain similar results.**

I. Introduction

The Air Traffic Management (ATM) system is a complex system composed of a large number of heterogeneous components, such as airports, aircraft, navigation systems, flight management systems (FMS), traffic controllers, and weather (see Kim et al. [1]). Therefore, its performance is affected by numerous factors. Within the trajectory-based-operations concept of SESAR and NextGen, aircraft trajectories are key to study ATM operations, which are subject to many uncertainties. Sources of uncertainty for aircraft trajectories include wind and severe weather, navigational errors, aircraft performance inaccuracies, or errors in the FMS, among others. The analysis of the impact of uncertainties in aircraft trajectories and its propagation through the flight segments is of great interest, since it might help to understand how sensitive the system is to the lack of precise data and measurement errors, and, therefore, aid in the design of a more robust ATM system, with improved safety levels.

Among those sources, weather uncertainty has perhaps the greatest impact on ATM operations, being responsible for much of the delays. Its analysis has been addressed by many authors, using different methods. For instance, Nilim et al. [2] consider a trajectory-based air traffic management scenario to minimize delays under weather uncertainty, where the weather processes are modeled as stationary Markov chains. Pepper et al. [3] present a method, based on Bayesian decision networks, for taking into account uncertain weather information in air traffic flow management. Clarke et al. [4] develop a methodology to study airspace capacity in the presence of weather uncertainty and formulates a stochastic dynamic programming algorithm for traffic flow management. Zheng and Zhao [5] develop a statistical model of wind uncertainties and apply it to stochastic trajectory prediction in the case of straight, level flight trajectories.

The framework for this work is the analysis of parametric uncertainties in aircraft trajectories, and, eventually, its effect on the ATM system. In this paper several tools are applied to study the effect of wind uncertainty during the cruise flight phase. In particular, fuel consumption is analyzed for a given cruise range under constant average wind. This study is relevant because wind is one of the main sources of uncertainty in trajectory prediction, and because cruise uncertainties have a large impact on the overall flight since the cruise phase is the largest portion of the flight (at least for long-haul routes).

Several methods have been proposed to study uncertainty propagation in dynamical systems. The easiest, but more expensive in computational terms, is the classical Monte-Carlo method (see for instance Thomopoulos [6]). Halder and Bhattacharya [7] classify those methods it two categories: parametric (in which one evolves the statistical moments) and non-parametric (in which the probability density function is evolved). They address the problem of uncertainty propagation in planetary entry, descent, and landing, using a non-parametric method that reduces to solving the stochastic Liouville equation. In this work, both parametric and non-parametric methods are applied.

One of the methods applied in this paper is the Generalized Polynomial Chaos (GPC) method (a parametric method according to Ref. [7]). The GPC representation was introduced by Wiener [8] and it is based on the fact that any second-order process (i.e., a process with finite second-order moments) can be represented as a Fourier-type series, with time-dependent coefficients, and using orthogonal polynomials as GPC basis functions in terms of random variables. A general introduction to GPC can be found in Xiu and Karniadakis [9] and in Schoutens [10], whereas details in numerical computations are studied in Debusschere et al. [11]. The method of polynomial chaos is used in the works of Prabhakar et al. [12] and Dutta and Bhattacharya [13] to study, respectively, uncertainty propagation and trajectory estimation, for hypersonic flight dynamics with uncertain initial data, by Fisher and Bhattacharya [14] and Okamoto and Tsuchiya [15] in the problem of optimal trajectory generation in the context of stochastic optimal control, by Jones et al. [16] for nonlinear propagation of orbits, and by Li et al. [17] in robust aircraft trajectory optimization.

The probability density function of the aircraft fuel mass is analyzed using an approximate non-parametric method developed in Vazquez and Rivas [18] to study uncertainty propagation for a cruise flight with probabilistic initial mass. The method is based on the resolution of the variational equation for the sensitivity function with respect to the initial condition.

The results are compared with the probability density function obtained by analytically solving the mass equation (to show the exactness of the methods), and with Monte Carlo simulations, which, as expected, require much larger computation times to obtain comparable results.

II. Mass evolution in cruise flight with constant average wind

To study the evolution of aircraft mass in cruise flight, the equations of flight mechanics for flight in a vertical plane (constant heading) are considered, under the following hypothesis: symmetric flight, flat Earth model, constant altitude, and constant velocity. Then, the equation of mass evolution is (taking into account the equation $T = D$, T being the thrust)

$$\dot{m} = -cD \tag{1}$$

where m is the aircraft mass, D the aerodynamic drag, and c the specific fuel consumption, which is considered constant under the previously stated hypothesis.

In this paper, the cruise range x_f and the final aircraft mass m_f are given (fixing m_f, instead of m_i, is consistent with having a fixed landing weight).

The drag can be written as $D = \frac{1}{2}\rho V^2 S C_D$, where ρ is the density, V the velocity, S the wing surface area, and the drag coefficient C_D is modeled by a parabolic polar of constant coefficients $C_D = C_{D_0} + kC_L^2$, where C_L is the lift coefficient given by $C_L = \frac{L}{\frac{1}{2}\rho V^2 S} = \frac{mg}{\frac{1}{2}\rho V^2 S}$, where the equation of motion $L = mg$ (g is the acceleration of gravity) has been used.

Using these definitions, an autonomous equation for mass evolution is obtained:

$$\dot{m} = -c\left(\frac{1}{2}\rho V^2 S C_{D_0} + m^2 \frac{kg^2}{\frac{1}{2}\rho V^2 S}\right) \tag{2}$$

Thus, one can write

$$\dot{m} = -(A + Bm^2) \tag{3}$$

where the constants A and B are defined as $A = \frac{c}{2}\rho V^2 S C_{D_0}$ and $B = \frac{2ckg^2}{\rho V^2 S}$. Note that $A, B > 0$. Equation (3) is a nonlinear equation describing the evolution of mass during cruise flight as a function of time.

To find the evolution of mass *as a function of distance*, consider the kinematic equation

$$\dot{x} = V + w \tag{4}$$

where x is the horizontal distance, and w is the average wind speed, considered constant. Combining (3) and (4) to eliminate time, one reaches

$$\frac{dm}{dx} = -\frac{A + Bm^2}{V + w} \tag{5}$$

which is to be solved *backwards* with the boundary condition

$$m(x_f) = m_f \tag{6}$$

To emphasize the dependence of the mass $m(x)$ on the wind, the mass is written as $m(x; w)$, even though often it is just denoted as m for the sake of simplicity.

Once the aircraft mass is obtained, the cruise fuel load follows from

$$m_F(w) = m(0; w) - m_f \tag{7}$$

III. Probabilistic wind model

In this paper the case where w is distributed as a uniform continuous variable is considered, see Figure 1. If one had more information on the wind uncertainty, a different distribution could be used. In the following, results are presented for the case where the mean of w is $\bar{w} = \mathrm{E}[w] = 0$, where $\mathrm{E}[\cdot]$ is the mathematical expectation. Hence, the probability density function is

$$f_w(w) = \begin{cases} \frac{1}{2\delta_w}, & w \in [-\delta_w, \delta_w] \\ 0, & w \notin [-\delta_w, \delta_w] \end{cases} \tag{8}$$

where δ_w is the width of the uniform distribution. The variance of w is

$$\mathrm{Var}[w] = \mathrm{E}[w^2] - (\mathrm{E}[w])^2 = \frac{\delta_w^2}{3}$$

Denoting by Δ the standardized uniform distribution taking values in the interval $[-1, 1]$, one has

$$w = \delta_w \Delta \tag{9}$$

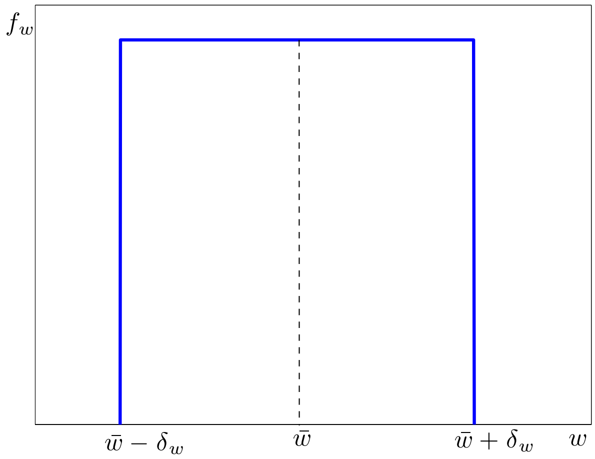

Fig. 1. Wind uniform distribution, with mean $\bar{w}$ and width δ_w.

IV. Analysis of fuel mass uncertainty

If the average wind w is uncertain, then the evolution of mass with time or distance is uncertain. In particular, since the aircraft final mass is fixed for a given cruise range, then the required fuel mass is uncertain. Note that, in such a case, the solution of Eqs. (5) and (6) is still valid but in a probabilistic sense, i.e., $m(x; w)$ is a random process.

In the following, for the specific probabilistic wind model of Section III, the resulting fuel mass is studied with several methods. The probability density function and the statistical properties of the distribution (mean, variance, typical deviation) are analyzed.

A. *Mean and variance of fuel mass*

To compute the mean and variance of the fuel mass, the Generalized Polynomial Chaos (GPC) method is used (see Ref. [8]), in which the process is represented as a Fourier-type series, with time-dependent coefficients, and orthogonal polynomials in terms of random variables are used as basis functions. The orthogonal polynomials used in GPC are chosen from the Askey scheme (a way of organizing certain orthogonal polynomials into a hierarchy, see Ref. [19]). If one chooses a family of polynomials which are orthogonal the convergence of the series is exponential. The orthogonality property implies that, when taking expectation with respect to the random variable for two polynomials of the family ϕ_i and ϕ_j, then $\mathrm{E}[\phi_i\phi_j] = \delta_{ij}\mathrm{E}[\phi_i^2]$, where δ_{ij} is the Kronecker's delta function. For the uniform distribution Δ, the adequate orthogonal polynomials are the Legendre polynomials $L_n(\Delta)$.

To apply the GPC method, one first write the final aircraft mass $m(x_f)$ in terms of the orthogonal polynomials. Since it is just a nonrandom constant, one can write

$$m(x_f) = m_f L_0(\Delta) \tag{10}$$

because $L_0(\Delta) = 1$. On the other hand, writing the random average wind (9) in terms of the orthogonal polynomials renders

$$w = \delta_w L_1(\Delta) \tag{11}$$

because $L_1(\Delta) = \Delta$.

Next, the aircraft mass $m(x;w)$ is written as

$$m(x;w) = \sum_{i=0}^{P} h_i(x) L_i(\Delta) \tag{12}$$

where the coefficients h_i are to be found using the mass equation (5), and P is the order of the approximation, which is to be taken sufficiently large. The advantage of the GPC method is that a small or moderate value of P is enough to get good results, thus resulting in a method that is not very intensive computationally. The process to obtain the coefficients h_i is presented in Appendix A.

Once the coefficients $h_i(0)$ are found, one can compute from Eq. (12) approximate values of the fuel mass mean and variance, as follows. For the mean one has

$$\begin{aligned}
\mathrm{E}[m(0;w)] &= \sum_{i=0}^{P} h_i(0)\mathrm{E}[L_i(\Delta)] \\
&= \sum_{i=0}^{P} h_i(0)\mathrm{E}[L_i(\Delta)L_0(\Delta)] \\
&= h_0(0)\mathrm{E}[L_0^2(\Delta)] \\
&= h_0(0)
\end{aligned} \tag{13}$$

thus,

$$\mathrm{E}[m_F] = h_0(0) - m_f$$

and for the variance

$$\begin{aligned}
\mathrm{Var}[m_F] &= \mathrm{Var}[m(0;w)] \\
&= \mathrm{E}[m^2(0;w)] - \mathrm{E}[m(0;w)]^2 \\
&= \sum_{i=0}^{P}\sum_{j=0}^{P} h_i(0)h_j(0)\mathrm{E}[L_i(\Delta)L_j(\Delta)] - h_0(0)^2 \\
&= \sum_{i=1}^{P} h_i^2(0)\mathrm{E}[L_i^2(\Delta)]
\end{aligned} \tag{14}$$

B. *Distribution function of fuel mass*

Next, an approximate method that is able to find the fuel mass probability density function $f_{m_F}(m_F)$ is considered. The method was developed in Vazquez and Rivas [18] to study the evolution of the aircraft mass probability density function in cruise flight with uncertain initial mass.

Recall that, given a random variable x with probability density function $f_x(x)$, if one defines another random variable y using a transformation g such that $y = g(x)$, then it is known that the probability density function $f_y(y)$ of y is given by (see Canavos [20])

$$f_y(y) = \frac{f_x(g^{-1}(y))}{|g'(g^{-1}(y))|} \tag{15}$$

expression which is valid only if the function $g(x)$ is invertible on the domain of x.

In this problem,

$$f_{m_F}(m_F) = \frac{f_w(g^{-1}(m_F))}{|g'(g^{-1}(m_F))|} \tag{16}$$

where

$$m_F = g(w) = m(0;w) - m_f \tag{17}$$

Thus, the analysis is valid only if the function $m_F = g(w)$ is invertible on the domain of w, that is, only if for two different values of wind w_1 and w_2, the required fuel masses m_{F_1} and m_{F_2} are different, which in this problem is obvious.

The idea of the method is to numerically approximate equation (16). For that, take n consecutive points from the domain of w, denoted as w^i, $i = 1, \ldots, n$, so that $w^1 < w^2 < \ldots < w^n$. Now, solving backwards the mass equation (5) for each i with parameter w^i and m_f as final condition, one can compute the value of aircraft mass at $x = 0$, $m^i(0) = m(0;w^i)$, and then

$$m_F^i = m(0;w^i) - m_f \tag{18}$$

The numerator of (16) is computed for each i as $f_w(w^i)$. To compute the denominator of (16), the function $g'(w)$ is needed; this function is obtained in terms of

$$\phi(x;w) \equiv \frac{\partial m(x;w)}{\partial w} \tag{19}$$

which is the sensitivity function of the solution m with respect to the parameter w. The process to obtain $\phi(x;w)$ is presented in Appendix B.

Once the function $\phi(x;w)$is found, one has the values of the probability density function $f_{m_F}(m_F)$ at the n points m_F^i, $i=1,\ldots,n$, as

$$f_{m_F}(m_F^i) = \frac{f_w(w^i)}{|\phi(0;w^i)|} \tag{20}$$

Hence, the mean and the typical deviation can be computed from

$$E[m_F] = \int_0^\infty m_F f_{m_F}(m_F) dm_F \tag{21}$$

$$(\sigma[m_F])^2 = \int_0^\infty m_F^2 f_{m_F}(m_F) dm_F - (E[m_F])^2 \tag{22}$$

V. Monte Carlo method

Now, in this section, the Monte Carlo method is applied to the problem considered in this paper, as follows. From the wind distribution randomly generate N samples $\{w_i\}$, $i=1,\ldots,N$, and use each of these samples to solve the mass equation (5), finding $m(0;w_i)$. Hence, one obtains a sequence of N samples $m_F^i = m(0;w_i) - m_f$, for $i=1,\ldots,N$. From these values one can directly find approximate values for the mean and the typical deviation. One can also obtain an approximate probability density function, as follows.

Choose n_F equidistant points $m_{F,j}$ to discretize the domain of the probability density function by setting

$$m_{F,1} = \min\{m_F^i\} \tag{23}$$

$$m_{F,n_F} = \max\{m_F^i\} \tag{24}$$

$$m_{F,j} = m_{F,1} + (j-1)d_F, \quad j=2,\ldots,n_F-1 \tag{25}$$

where

$$d_F = \frac{m_{F,n_F} - m_{F,1}}{n_F - 1} \tag{26}$$

is the distance between discretization points.

Then, for $j=2,\ldots,n_F-1$, the probability density function at the discretization points is approximated as follows

$$f_{m_F}(m_{F,j}) = \frac{N_j}{N} d_F \tag{27}$$

where N_j is the number of samples m_F^i satisfying

$$m_F^i \in \left[m_{F,j} - \frac{d_F}{2}, m_{F,j} + \frac{d_F}{2}\right] \tag{28}$$

i.e. the number of samples which are closer to $m_{F,j}$. At the end points there is a slightly different definition

$$f_{m_F}(m_{F,1}) = \frac{N_1}{2N} d_F \tag{29}$$

$$f_{m_F}(m_{F,n_F}) = \frac{N_{n_F}}{2N} d_F \tag{30}$$

where N_1 is defined as the number of samples m_F^i satisfying

$$m_F^i \in \left[m_{F,1}, m_{F,1} + \frac{d_F}{2}\right] \tag{31}$$

and N_{n_F} is defined as the number of samples m_F^i satisfying

$$m_F^i \in \left[m_{F,n_F} - \frac{d_F}{2}, m_{F,n_F}\right] \tag{32}$$

The main problem of the method is that many samples (i.e. a large value of N) are required, and for each sample, the differential equation (5) has to be solved. Since each run of the differential equation is independent of others, this task is frequently parallelized to reduce the computational load.

Regarding the quality of the estimates, Bayer et al. [21] offer a formula that allows to compute the probability that the error of the mean obtained by the Monte Carlo method is larger than some given tolerance. Calling $\bar{m}_F$ the estimated mean, $E[m_F]$ the true mean, ε the tolerance, and $\sigma[m_F]$ the fuel mass typical deviation, one has

$$\Pr\left[|\bar{m}_F - E[m_F]| < \varepsilon\right] \approx 2\left(1 - \Phi\left[\frac{\sqrt{N}\varepsilon}{\sigma[m_F]}\right]\right) \tag{33}$$

where Φ is standard normal cumulative distribution function. This formula can be used to estimate the quality of the approximation.

VI. Results

Now results are presented for the following values of the different parameters: $C_{D_0} = 0.015$, $k = 0.042$, $\rho_0 = 1.225$ kg/m^3, $\rho = 0.5\rho_0$, $V = 200$ m/s, $c = 5\cdot 10^{-5}$ s/m, $S = 150$ m^2, $g = 9.8$ m/s^2, $m_f = 55000$ kg, and $x_f = 2500$ km. The value chosen for the uncertain wind width is $\delta_w = 50$ m/s (this value has been chosen deliberately large to magnify the uncertainty effects), hence the typical deviation is $\sigma[w] = \sqrt{\mathrm{Var}[w]} = \frac{\delta_w}{\sqrt{3}} = 28.87$ m/s. The computations were performed on a Mac laptop with a 2GHZ Intel Core i7 with 4GB of RAM using Matlab, without parallelization.

The resulting probability density function of fuel mass is shown in Figure 2. For the approximate method, the number of points taken is $n=1000$, value that has proven to be good enough. The numerical results totally agree with the exact results given in Appendix C.

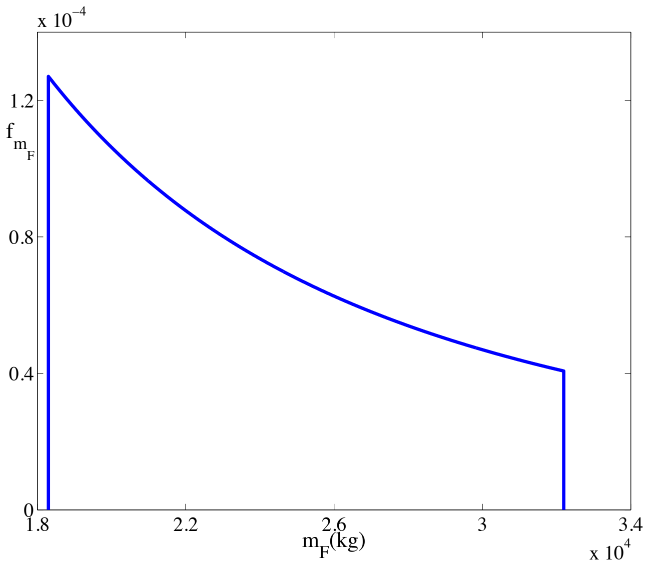

Fig. 2. Fuel mass density function, exact and computed by the approximate method. The result is indistinguishable.

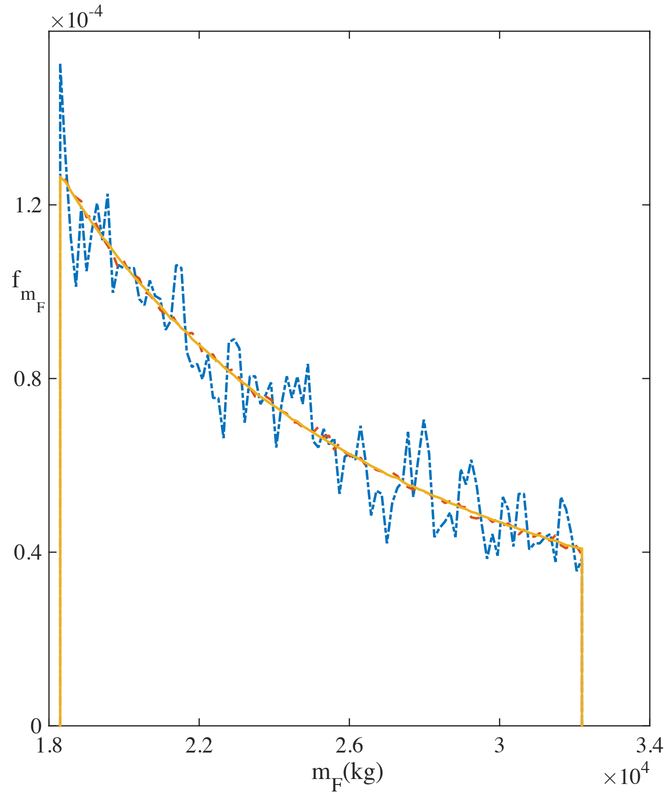

Fig. 3. Fuel mass density function computed with Monte Carlo method, for $N = 10^4$ (dashed), $N = 10^6$ (dash-dotted) and $N = 50 \cdot 10^6$ (solid).

In Figure 3, the results of the Monte Carlo simulations are presented. Three cases are shown, for $N = 10^4$, $N = 10^6$ and $N = 50 \cdot 10^6$, in all cases with $n_F = 100$. According to formula (33), these cases would have, when approximating the fuel mass mean, an approximate error (with 99% certainty) of, respectively, 100 kg, 10 kg and 1.5 kg. The approximate probability density function shape is captured in all cases, but only the larger samples obtain a good approximation. For $N = 50 \cdot 10^6$ the difference with the exact result is indistinguishable in the figure.

The results of fuel mass mean and typical deviation are given in Table I, along with the computation time of each method and the relative error compared with the exact results of Appendix C. For the GPC method an expansion with $P = 4$ terms is used, value that has proven to be good enough. It must be noted that the fastest method is GPC, whereas the most precise method is the approximate method. On the other hand, the Monte Carlo method, although easy to set up, is very inefficient and requires many runs (and therefore a much longer time) to obtain comparable results.

An interesting result is described next. If one computes the mass of fuel required for the average wind ($w = 0$), one obtains $m_F = 23320.6$ kg. This value is 621 kg smaller than the mean mass required for the considered distribution of wind (even though the wind distribution has zero mean). Mathematically, this can be expressed as

$$E[m_F(w)] \neq m_F(E[w])$$

This result is due to the nonlinearity of the relationship between m_F and w, as given in Appendix C. This implies that, when many flights are considered, due to the presence of the uncertain wind the overall fuel consumption would be larger than the fuel consumption corresponding to the expected average wind speed.

VII. Conclusions

The problem of fuel consumption in cruise flight subject to an uncertain average wind has been studied, using a nonlinear model which has known analytical solution. The average wind has been modeled as a random variable with uniform distribution function. Even though the model is quite simple, it has been shown that it yields very interesting results.

To study the distribution function of the fuel mass, two methods have been considered: an approximate method developed by the authors and the Monte Carlo method. The approximate method is applicable to problems in which there is just one random variable and for the analysis of distribution functions of functions of the random variable which are invertible. The results obtained with this method have been compared with the exact analytical results, showing an excellent agreement in all cases; thus, the accuracy of the method has been assessed. For the Monte Carlo method, it has been shown that it requires much larger computation times to obtain comparable results.

To obtain the mean and variance of the fuel mass, in addition to the previous methods, the generalized polynomial chaos (GPC) method has been also used, where an expansion with just four terms has proven to be accurate enough.

An important conclusion of this study is that, even though the mean of the considered wind distribution function is zero, the mean of the required fuel mass is considerable larger that the fuel mass for zero wind. The stochastic methodology presented is able to quantitatively estimate such increase.

The general framework for this paper is the development of a methodology to manage weather uncertainty suitable to be integrated into the trajectory planning process. This work is a first step that has focussed on the assessment of the impact of wind uncertainty on aircraft trajectory, and in particular on the cruise fuel load.

The methods presented in this paper can be applied to other flight phases defined by more complicated flight conditions, and they can be extended to consider other sources of uncertainty or other models of uncertain wind. The analysis of these problems is left for future work.

Acknowledgments

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TABLE I
VALUES OF MEAN AND TYPICAL DEVIATION OF FUEL MASS

Method	Computation Time	E$[m_F)]$ (kg)	Mean relative error(%)	$\sigma[m_F]$ (kg)	Typical deviation relative error (%)
Exact		23941.7		3924.9	
GPC ($P = 4$)	0.15 s	23941.7	10^{-5}	3924.9	10^{-4}
Approximate ($n = 1000$)	4 s	23941.7	10^{-10}	3924.9	$2 \cdot 10^{-10}$
Monte Carlo ($N = 10^4$)	31.3 s	23978.6	0.15	3933.4	0.21
Monte Carlo ($N = 10^6$)	≈ 50 min	23938.6	0.013	3917.7	0.18
Monte Carlo ($N = 50 \cdot 10^6$)	≈ 45 h	23941.5	$7 \cdot 10^{-4}$	3925.3	0.011

APPENDIX A: GPC COEFFICIENTS

Substituting Eqs. (11) and (12) in Eq. (5), the following equation is obtained

$$\sum_{i=0}^{P} \dot{h}_i(x) L_i(\Delta) = -\frac{A + B\sum_{i,j=0}^{P} h_i(x)h_j(x)L_i(\Delta)L_j(\Delta)}{V + \delta_w L_1(\Delta)} \quad (34)$$

To write (34) in a more standard form, first, cross-multiply (34) by $V + \delta_w L_1(\Delta)$, then, multiply it by $L_l(\Delta)$ for $l = 0, \dots, P$, and take expectation with respect to Δ, and using the orthogonality property of the L_l polynomials, one obtains $P + 1$ equations

$$\begin{aligned} & V\dot{h}_l(x)E[L_l^2(\Delta)] + \delta_w \sum_{i=0}^{P} \dot{h}_i(x)E[L_i(\Delta)L_l(\Delta)L_1(\Delta)] \\ = \; & -A\delta_{0l} - B\sum_{i=0}^{P}\sum_{j=0}^{P} h_i(x)h_j(t)E[L_i(\Delta)L_j(\Delta)L_l(\Delta)] \end{aligned}$$

for $l = 0, \dots, P$. Calling $C_{ijl} = \frac{E\left[L_i L_j L_l\right]}{E\left[L_l^2\right]}$ (which is a number that can be exactly computed since the involved expectations are just integrals of polynomials) it follows that

$$V\dot{h}_l + \delta_w \sum_{i=0}^{P} \dot{h}_i(x) C_{i1l} = -A\delta_{0l} - B\sum_{i=0}^{P}\sum_{j=0}^{P} h_i h_j C_{ijl} \quad (35)$$

for $l = 0, \dots, P$. The right-hand side of (35) is a non-linear function of all coeficients h_i, which is denoted as $f_l(h_0, \dots, h_P)$. Defining the vectors

$$\vec{h}(x) = \begin{bmatrix} h_0(x) \\ h_1(x) \\ \vdots \\ h_P(x) \end{bmatrix}, \quad \vec{f}(\vec{h}) = \begin{bmatrix} f_0(\vec{h}) \\ f_1(\vec{h}) \\ \vdots \\ f_P(\vec{h}) \end{bmatrix} \quad (36)$$

and defining the matrix $\mathbf{A}$ as

$$A_{li} = V\delta_{li} + \delta_w C_{i1l}$$

Eq. (35) can be written as

$$\frac{d}{dx}\vec{h}(x) = \mathbf{A}^{-1}\vec{f}(\vec{h}(x)) \quad (37)$$

which is a system of $P + 1$ nonlinear coupled ordinary differential equations written in standard form. To find the coefficients at $x = 0$, Eq. (37) needs to be solved backwards from the final condition

$$h_0(x_f) = m_f, \quad h_l(x_f) = 0, \quad \text{for } l = 1, \dots, P \quad (38)$$

up to $x = 0$.

APPENDIX B: MASS SENSITIVITY FUNCTION

According to Eq. (17)

$$g'(w) = \frac{\partial m(0; w)}{\partial w} = \phi(0; w) \quad (39)$$

Now the sensitivity function $\phi(x; w)$ is obtained as the solution of the following differential equation

$$\begin{aligned} \frac{d}{dx}\phi(x; w) &= \frac{d}{dx}\left(\frac{\partial m}{\partial w}\right) \\ &= \frac{A + Bm^2}{(V + w)^2} - \frac{2Bm}{V + w}\frac{\partial m}{\partial w} \\ &= \frac{A + Bm^2}{(V + w)^2} - \frac{2Bm}{V + w}\phi(x; w) \end{aligned} \quad (40)$$

with final condition

$$\phi(x_f; w) = \frac{\partial m(x_f; w)}{\partial w} = \frac{\partial m_f}{\partial w} = 0 \quad (41)$$

where $m = m(x; w)$ is the solution of Eqs. (5) and (6). This is to be solved numerically. Once the solution is found, one gets $\phi(0; w)$. Therefore, the denominator of (16) is computed for each i as

$$|g'(w^i)| = |\phi(0; w^i)| \quad (42)$$

APPENDIX C: EXACT SOLUTION

The cruise fuel load is

$$m_F = \frac{\left(m_f^2 + \frac{A}{B}\right)\tan\left(\frac{\sqrt{AB}x_f}{V + w}\right)}{\sqrt{\frac{A}{B}} - m_f \tan\left(\frac{\sqrt{AB}x_f}{V + w}\right)} \quad (43)$$

and the fuel mass probability density function is

$$f_{m_F}(m_F) = \begin{cases} G(m_F), & m_F \in [m_{F_1}, m_{F_2}] \\ 0, & m_F \notin [m_{F_1}, m_{F_2}] \end{cases} \quad (44)$$

where

$$G(m_F) = \frac{Ax_f}{2\delta_w\left[(m_f+m_F)^2+\frac{A}{B}\right]} \times \left[\arctan\left(\frac{m_F\sqrt{\frac{A}{B}}}{m_f^2+\frac{A}{B}+m_F m_f}\right)\right]^{-2} \quad (45)$$

$$m_{F_1} = \frac{\left(m_f^2+\frac{A}{B}\right)\tan\left(\frac{\sqrt{AB}x_f}{V+\delta_w}\right)}{\sqrt{\frac{A}{B}}-m_f\tan\left(\frac{\sqrt{AB}x_f}{V+\delta_w}\right)} \quad (46)$$

$$m_{F_2} = \frac{\left(m_f^2+\frac{A}{B}\right)\tan\left(\frac{\sqrt{AB}x_f}{V-\delta_w}\right)}{\sqrt{\frac{A}{B}}-m_f\tan\left(\frac{\sqrt{AB}x_f}{V-\delta_w}\right)} \quad (47)$$

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